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Curvature dependent lower bounds for the first eigenvalue of the Dirac operator

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Abstract

Using Weitzenböck techniques on any compact Riemannian spin manifold we derive inequalities that involve a real parameter and join the eigenvalues of the Dirac operator with curvature terms. The discussion of these inequalities yields vanishing theorems for the kernel of the Dirac operator D and lower bounds for the spectrum of D^2 if the curvature satisfies certain conditions.

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1. Introduction

In 1980, Friedrich [1] proved that, on any compact Riemannian spin n -manifold M of scalar curvature S with $S_0 := \min\{S(x)|x \in M\} > 0$, every eigenvalue λ of the Dirac operator D satisfies the inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0. \quad (1)$$

In special geometric situations, better estimates are known (see [4,6]). For example, if M is a spin Kähler manifold of complex dimension m and scalar curvature $S > 0$, we have the inequalities

$$\lambda^2 \geq \begin{cases} \frac{m+1}{4m} S_0 & (m \text{ odd}), \\ \frac{m}{4(m-1)} S_0 & (m \text{ even}). \end{cases} \quad (2)$$

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The estimates (1), (2) are sharp in the sense that there are manifolds for which the given lower bound itself is an eigenvalue of D^2 . But this kind of estimate by the scalar curvature only is not useful if S has zeros or attains negative values. Hence, the question arises if there exist lower bounds for the spectrum of D^2 that depend on additional curvature terms. For certain manifolds whose curvature tensor or Weyl tensor, respectively, is divergence-free (co-closed and, hence, harmonic) such lower bounds have been obtained recently (see [2,3]). In the case of a compact Riemannian spin n -manifold M with divergence-free curvature tensor $R(\delta R = 0)$, scalar curvature $S = 0$, and nowhere vanishing Ricci tensor, for example, the estimate

$$\lambda^2 > \frac{1}{4} \frac{|\text{Ric}|_0^2}{|\kappa_0| + |\text{Ric}|_0 \sqrt{(n-1)/n}} \tag{3}$$

is valid, where $|\text{Ric}|_0 > 0$ denotes the minimum of the length of the Ricci tensor and κ_0 the smallest eigenvalue of Ric on M [2, Theorem 2.2]. Moreover, it has been proved that $\ker(D)$ is trivial, i.e., there are no harmonic spinors if M is compact with divergence-free curvature tensor and scalar curvature $S \leq 0$ such that the inequality

$$|\text{Ric}|_0^2 > S \cdot \kappa_0 \tag{4}$$

holds [2, Theorem 2.2]. We recall that S is constant here, since the supposition $\delta R = 0$ is equivalent to the symmetry property

$$(\nabla_X \text{Ric})Y = (\nabla_Y \text{Ric})X \tag{5}$$

of the covariant derivative ∇Ric of the Ricci tensor, which immediately implies $dS = 0$. A more general supposition than (5) is

$$(\nabla_X \text{Ric})Y - (\nabla_Y \text{Ric})X = \frac{1}{2(n-1)}(X(S)Y - Y(S)X). \tag{6}$$

For dimension $n \geq 4$, (6) is equivalent to the condition that the Weyl tensor W is divergence-free ($\delta W = 0$) and, hence, harmonic ($dW = 0, \delta W = 0$). In the compact conformally nonflat case with $\delta W = 0$, the estimate

$$\lambda^2 \geq \frac{1}{8(n-1)} \left((2n-1)S_0 + \sqrt{S_0^2 + \frac{n-1}{n} \left(\frac{4v_0}{\mu} \right)^2} \right) \tag{7}$$

was proved for any eigenvalue λ of the Dirac operator, where $v_0 \geq 0$ and $\mu > 0$ are conformal invariants depending on W only. For $S_0 > 0$, (7) yields a better estimate than (1) if $v_0 > 0$. For $S_0 \leq 0$, the lower bound in (7) is positive if $2v_0 > n\mu|S_0|$ [3, Theorem 3.1]. In this paper we prove estimates similar to (3) and (7) which, however, do not make use of the suppositions (5) or (6), respectively. Moreover, we obtain vanishing theorems for the space $\ker(D)$ of harmonic spinors which are generalizations of those in [2,3]. Our results are based on Weitzenböck formulas for modified twistor operators, which can partially be found in [2,3,5] already. However, what is new in this paper is the combination of the various Weitzenböck formulas for the modified twistor operators.

2. Curvature endomorphisms of the spinor bundle

Let M be any Riemannian spin n -manifold with Riemannian metric g and spinor bundle Σ . As usual, we denote by ∇ the covariant derivative induced by g on vector fields as well as on spinor fields (Levi-Civita connection). For any vector fields X, Y, Z and any spinor field ψ , the Riemannian curvature tensor R and the corresponding curvature tensor C of the spinor bundle are defined by

$$R(X, Y)Z := \nabla_{X,Y}^2 Z - \nabla_{Y,X}^2 Z, \quad C(X, Y)\psi := \nabla_{X,Y}^2 \psi - \nabla_{Y,X}^2 \psi,$$

where we use the notation

$$\nabla_{X,Y}^2 := \nabla_X \circ \nabla_Y - \nabla_{\nabla_X Y}$$

for the tensorial derivatives of second order. Given a local frame of vector fields (X_1, \dots, X_n) , we denote by (X^1, \dots, X^n) the associated coframe defined by $X^k := g^{kl} X_l$, where (g^{kl}) is the inverse of the matrix (g_{kl}) with $g_{kl} := g(X_k, X_l)$. Thus, for any orthonormal frame, we have $X^k = X_k$ ($k = 1, \dots, n$). Then the Ricci tensor Ric , the scalar curvature S , and the Dirac operator D are locally given by $\text{Ric}(X) = R(X, X_k)X^k$, $S = \text{tr}(\text{Ric}) = g(\text{Ric}(X_k), X^k)$ and $D\psi = X^k \cdot \nabla_{X_k} \psi$, respectively.

For the reader's convenience, we summarize some well-known, important identities:

$$C(X, Y) = \frac{1}{4} X_k \cdot R(X, Y)X^k, \tag{8}$$

$$X_k \cdot C(X^k, X) = \frac{1}{2} \text{Ric}(X) = C(X^k, X) \cdot X_k, \tag{9}$$

$$X_k \cdot \text{Ric}(X^k) = -S = \text{Ric}(X^k) \cdot X_k, \tag{10}$$

$$X^k \cdot \nabla_{X_k, X}^2 \psi = \nabla_X D\psi + \frac{1}{2} \text{Ric}(X) \cdot \psi, \tag{11}$$

$$X^k \cdot \nabla_{X, X_k}^2 \psi = \nabla_X D\psi. \tag{12}$$

The curvature endomorphism $C(X, Y)$ is anti-selfadjoint with respect to the Hermitian scalar product $\langle \cdot, \cdot \rangle$ on Σ , i.e., we have

$$C(X, Y)^* = -C(X, Y). \tag{13}$$

Thus, the endomorphism $C^2(X, Y) := C(Y, X_k) \circ C(X^k, X)$ has the property

$$C^2(X, Y)^* = C^2(Y, X) \tag{14}$$

and, hence, the endomorphism $G := C^2(X_k, X^k)$ of Σ is selfadjoint and nonnegative

$$G^* = G, \quad G \geq 0. \tag{15}$$

Let W denote the Weyl tensor of M and consider the curvature endomorphisms $B(X, Y) := (1/4)X_k \cdot W(X, Y)X^k$, $B^2(X, Y) := B(Y, X_k) \circ B(X^k, X)$, $H := B^2(X_k, X^k)$. Then we have analogously:

$$B(X, Y)^* = -B(X, Y), \quad B^2(X, Y)^* = B^2(Y, X), \tag{16}$$

$$X_k \cdot B(X^k, X) = 0 = B(X^k, X) \cdot X_k, \tag{17}$$

$$H^* = H, \quad H \geq 0. \tag{18}$$

The following lemma is proved by straightforward calculations.

Lemma 2.1. *The endomorphisms G and H are related by*

$$G = H + \frac{1}{8}(|R|^2 - |W|^2) = H + \frac{1}{2(n-2)} \left| \text{Ric} - \frac{S}{n} \right|^2 + \frac{S^2}{4n(n-1)}. \tag{19}$$

Moreover, if $H = H_0 + H_2 + H_4$ is the decomposition of H in the Clifford algebra into the components H_0, H_2, H_4 of degree 0, 2 and 4, respectively, then

$$H_0 = \frac{1}{8}|W|^2, \quad H_2 = 0. \tag{20}$$

Using the notations $\delta R(X) := (\nabla_{X_k} R)(X, X^k)$, $\delta C(X) := (\nabla_{X_k} C)(X, X^k)$ and $\delta W(X) := (\nabla_{X_k} W)(X, X^k)$, $\delta B(X) := (\nabla_{X_k} B)(X, X^k)$ we have the equations

$$\delta C(X) = \frac{1}{4} X_k \cdot \delta R(X) X^k, \quad \delta B(X) = \frac{1}{4} X_k \cdot \delta W(X) X^k. \tag{21}$$

Moreover, it holds that

$$\delta B(X) = \delta C(X) + \frac{1}{8(n-1)} (X \cdot dS - dS \cdot X). \tag{22}$$

The second Bianchi identity implies

$$g(\delta R(X)Y, Z) = g((\nabla_Y \text{Ric})Z - (\nabla_Z \text{Ric})Y, X). \tag{23}$$

Inserting this into (21) we obtain

$$\delta C(X) = \frac{1}{4} (X^k \cdot (\nabla_{X_k} \text{Ric})X - (\nabla_{X_k} \text{Ric})X \cdot X^k). \tag{24}$$

Using (21) and (24) we find the identities

$$X_k \cdot \delta C(X^k) = \frac{1}{4} dS, \quad X_k \cdot \delta B(X^k) = 0. \tag{25}$$

For any vector field X , the endomorphisms $\delta C(X)$ and $\delta B(X)$ of Σ are antiselfadjoint

$$\delta C(X)^* = -\delta C(X), \quad \delta B(X)^* = -\delta B(X). \tag{26}$$

Thus, the endomorphisms $E := -\delta C(X_k) \circ \delta C(X^k)$ and $F := -\delta B(X_k) \circ \delta B(X^k)$ are selfadjoint and nonnegative

$$E^* = E, \quad E \geq 0, \quad F^* = F, \quad F \geq 0. \tag{27}$$

By (22) and (25), we obtain

$$E = F + \frac{1}{16(n-1)} |dS|^2. \tag{28}$$

Moreover, by Proposition 3.1. in [5], it holds that

$$E = \frac{1}{4} |\nabla \text{Ric}|^2 - \frac{1}{16} |dS|^2 + \frac{1}{8} [\nabla_{X_j} \text{Ric}, \nabla_{X_k} \text{Ric}](X_j) \cdot X^j \cdot X^k \cdot X^l, \tag{29}$$

where $[\cdot, \cdot]$ denotes the commutator of endomorphisms. Now we introduce some numbers that occur in our following eigenvalue estimates. Let M be compact. We denote by ν_0 the infimum of all eigenvalues of H on M . By definition, ν_0 is a conformal invariant and we have the inequality

$$\nu_0 |\psi|^2 \leq \langle H\psi, \psi \rangle \tag{30}$$

for any $\psi \in \Gamma(\Sigma)$. By (19) we see that Ric and ν_0 are obstructions against the existence of parallel spinors since $\nabla\psi = 0$ implies $C(X, Y) \cdot \psi = 0$ for all vector fields X, Y and, hence, $G\psi = 0$. The Schrödinger–Lichnerowicz formula

$$\nabla^* \nabla = D^2 - \frac{1}{4} S \tag{31}$$

shows that, in the compact case with vanishing scalar curvature, any harmonic spinor ψ ($D\psi = 0$) is parallel. Hence, $\ker(D) = 0$ follows if M is compact and Ricci flat, but $\nu_0 > 0$. In special situations, ν_0 can easily be computed [3, Section 3]. Further, we consider the number

$$\mu := \sup\{\|B(X, Y)\| \mid x \in M, X, Y \in T_x M, g(X, Y) = 0, |X| = |Y| = 1\},$$

where $\|\cdot\|$ denotes the operator norm. By definition, $\mu \geq 0$ is a conformal invariant. By ζ we denote the corresponding supremum if B is replaced by the spin curvature tensor C .

Lemma 2.2. *For any $\psi \in \Gamma(\Sigma)$, the inequalities*

$$|\langle C^2(X^k, X^l) \cdot \nabla_{X_k} \psi, \nabla_{X_l} \psi \rangle| \leq (n - 1)^2 \zeta^2 |\nabla\psi|^2, \tag{32}$$

$$|\langle B^2(X^k, X^l) \cdot \nabla_{X_k} \psi, \nabla_{X_l} \psi \rangle| \leq (n - 1)^2 \mu^2 |\nabla\psi|^2 \tag{33}$$

are valid.

Proof. Let (X_1, \dots, X_n) be any local orthonormal frame. Then, for all $k, l \in \{1, \dots, n\}$, we have the estimate

$$\sum_{j=1}^n \|C(X_j, X_k)\| \|C(X_j, X_l)\| \leq \begin{cases} (n - 1)\zeta^2 & \text{if } k = l, \\ (n - 2)\zeta^2 & \text{if } k \neq l. \end{cases} \tag{*}$$

Now it holds that

$$\begin{aligned} & |\langle C^2(X^k, X^l) \nabla_{X_k} \psi, \nabla_{X_l} \psi \rangle| \\ & \leq \sum_{j,k,l} |C(X_j, X_k) \nabla_{X_k} \psi, C(X_j, X_l) \nabla_{X_l} \psi| \\ & \leq \sum_{j,k,l} |C(X_j, X_k) \nabla_{X_k} \psi| |C(X_j, X_l) \nabla_{X_l} \psi| \\ & \leq \sum_{j,k,l} \|C(X_j, X_k)\| \|C(X_j, X_l)\| \|\nabla_{X_k} \psi\| \|\nabla_{X_l} \psi\| \\ & = \sum_{j,k} \|C(X_j, X_k)\|^2 |\nabla_{X_k} \psi|^2 + \sum_{j,k \neq l} \|C(X_j, X_k)\| \|C(X_j, X_l)\| \|\nabla_{X_k} \psi\| \|\nabla_{X_l} \psi\| \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(*)}{\leq} (n-1)\zeta^2 |\nabla\psi|^2 + (n-2)\zeta^2 \sum_{k \neq l} |\nabla_{X_k}\psi| |\nabla_{X_l}\psi| \\
 &= \zeta^2 |\nabla\psi|^2 + (n-2)\zeta^2 \sum_{k,l} |\nabla_{X_k}\psi| |\nabla_{X_l}\psi| \\
 &= \zeta^2 |\nabla\psi|^2 + (n-2)\zeta^2 \left(\sum_k |\nabla_{X_k}\psi| \right)^2 \\
 &\leq \zeta^2 |\nabla\psi|^2 + n(n-2)\zeta^2 \sum_k |\nabla_{X_k}\psi|^2 = (n-1)^2 \zeta^2 |\nabla\psi|^2.
 \end{aligned}$$

This proves (32). An analogous calculation yields (33). □

We remark that (33) is a better estimate than the corresponding estimate (23) in [3].

3. Estimates depending on the Ricci tensor

Let M be a Riemannian spin n -manifold and

$$\mathcal{D} : \Gamma(\Sigma) \rightarrow \Gamma(TM \otimes \Sigma)$$

the corresponding twistor operator locally given by $\mathcal{D}\psi := X^k \otimes \mathcal{D}_{X_k}\psi$ with

$$\mathcal{D}_X\psi := \nabla_X\psi + \frac{1}{n}X \cdot D\psi.$$

For $s, t \in \mathbb{R}$, we consider the differential operators of first order (modified twistor operators)

$$\mathcal{P}^s, \mathcal{Q}^t : \Gamma(\Sigma) \rightarrow \Gamma(TM \otimes \Sigma)$$

defined by $\mathcal{P}^s\psi := X^k \otimes \mathcal{P}_{X_k}^s\psi$, $\mathcal{Q}^t\psi := X^k \otimes \mathcal{Q}_{X_k}^t\psi$ and

$$\begin{aligned}
 \mathcal{P}_X^s\psi &:= \mathcal{D}_X\psi - s \left(\delta C(X) + \frac{1}{4n}X \cdot dS \right) \cdot \psi, \\
 \mathcal{Q}_X^t &:= \mathcal{D}_X\psi + t \left(\text{Ric} - \frac{S}{n} \right) (X) \cdot D\psi.
 \end{aligned}$$

The image of \mathcal{D} is contained in the kernel of the Clifford multiplication, i.e.,

$$X^k \cdot \mathcal{D}_{X_k}\psi = 0 \tag{34}$$

for all $\psi \in \Gamma(\Sigma)$. Thus, by (10) and (25), we see that the images of \mathcal{P}^s and \mathcal{Q}^t are also contained in the kernel of the Clifford multiplication

$$X^k \cdot \mathcal{P}_{X_k}^s\psi = 0, \quad X^k \cdot \mathcal{Q}_{X_k}^t\psi = 0. \tag{35}$$

For any $\psi \in \Gamma(\Sigma)$, one has the well-known formula

$$|\mathcal{D}\psi|^2 = |\nabla\psi|^2 - \frac{1}{n}|D\psi|^2. \tag{36}$$

We introduce the selfadjoint nonnegative endomorphism

$$\mathcal{E} := E - \frac{1}{16n} |dS|^2 \stackrel{(28)}{=} F + \frac{1}{16n(n-1)} |dS|^2$$

and by straightforward calculations we obtain

$$|\mathcal{P}^s \psi|^2 = |D\psi|^2 + 2s \operatorname{Re}(\delta C(X^k) \nabla_{X_k} \psi, \psi) + \frac{s}{2n} \operatorname{Re}(D\psi, dS \cdot \psi) + s^2 \langle \mathcal{E}\psi, \psi \rangle, \tag{37}$$

$$|\mathcal{Q}^t \psi|^2 = |D\psi|^2 - 2t \operatorname{Re}(\operatorname{Ric}(X^k) \nabla_{X_k} \psi, D\psi) + 2t \frac{S}{n} |D\psi|^2 + t^2 \left| \operatorname{Ric} - \frac{S}{n} \right|^2 |D\psi|^2. \tag{38}$$

Lemma 3.1. *Let λ be any eigenvalue of the Dirac operator D . Then, for all corresponding eigenspinors ψ ($D\psi = \lambda\psi$), it holds that*

$$\begin{aligned} & \frac{1}{2} (|\mathcal{P}^s \psi|^2 + |\mathcal{Q}^t \psi|^2) \\ &= |D\psi|^2 + t \frac{S}{n} \lambda^2 |\psi|^2 - t \left(\left(\lambda^2 - \frac{S}{4} \right) \left(|\nabla \psi|^2 - \left(\lambda^2 - \frac{S}{4} \right) |\psi|^2 \right) \right. \\ & \quad \left. + \frac{1}{4} |\operatorname{Ric}|^2 |\psi|^2 + \langle \nabla_{\operatorname{Ric}(X_k)} \psi, \nabla_{X^k} \psi \rangle \right) + t \operatorname{div}(X_\psi) \\ & \quad + \frac{t^2}{2} \left(\langle \mathcal{E}\psi, \psi \rangle + \lambda^2 \left| \operatorname{Ric} - \frac{S}{n} \right|^2 |\psi|^2 \right), \end{aligned} \tag{39}$$

where X_ψ is the vector field locally defined by

$$X_\psi := \operatorname{Re}(\langle (D^2 - \frac{1}{4}S)\psi, \nabla_{X^k} \psi \rangle + \langle \nabla_{X_j} D\psi + \frac{1}{2} \operatorname{Ric}(X_j) \cdot \psi, X^k \cdot \nabla_{X^j} \psi \rangle) X_k.$$

Proof. By Lemma 1.4 in [2] and (24), for all $\psi \in \Gamma(S)$, we have the identity

$$\begin{aligned} & \operatorname{Re} \langle \operatorname{Ric}(X^k) \nabla_{X_k} D\psi, \psi \rangle - \operatorname{Re} \langle \delta C(X^k) \nabla_{X_k} \psi, \psi \rangle \\ &= |\nabla D\psi|^2 - |D^2 - \frac{1}{4}S \rangle \psi|^2 - \frac{1}{4} S |\nabla \psi|^2 + \frac{1}{4} |\operatorname{Ric}|^2 |\psi|^2 \\ & \quad + \langle \nabla_{\operatorname{Ric}(X^k)} \psi, \nabla_{X^k} \psi \rangle - \operatorname{div}(X_\psi). \end{aligned} \tag{40}$$

Using (37), (38) and (40) we obtain (39). □

Now, for M being compact, let ϑ denote the supremum of all eigenvalues of \mathcal{E} on Σ . Then $\vartheta \geq 0$ and

$$\langle \mathcal{E}\psi, \psi \rangle \leq \vartheta |\psi|^2 \tag{41}$$

for any $\psi \in \Gamma(\Sigma)$. Moreover, let κ_0 be the infimum of all eigenvalues of Ric on TM and let κ denote the supremum of its eigenvalues. Then, for any $\psi \in \Gamma(\Sigma)$, the inequalities

$$\kappa_0 |\nabla \psi|^2 \leq \langle \nabla_{\text{Ric}(X_k)} \psi, \nabla_{X^k} \psi \rangle \leq \kappa |\nabla \psi|^2 \tag{42}$$

are valid. We denote by S_0 the minimum of the scalar curvature S and by S_1 its maximum and we use the notation

$$S_* := \begin{cases} S_0 & \text{if } \kappa_0 \leq 0, \\ S_1 & \text{if } \kappa_0 > 0. \end{cases}$$

Further, we introduce the functions $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \alpha(t) &:= 1 + \frac{nt}{n-1} \left(\frac{S_1}{n} - \kappa_0 + \frac{S_1 - S_0}{4} \right) + \frac{nt^2}{2(n-1)} \left| \text{Ric} - \frac{S}{n} \right|_1^2, \\ \beta(t) &:= S_0 + t \left(|\text{Ric}|_0^2 - S_* \kappa_0 + \frac{S_0(S_1 - S_0)}{4} \right) - 2\vartheta t^2, \end{aligned}$$

where $|\text{Ric}|_0$ denotes the minimum of the function $|\text{Ric}|$ and $|\text{Ric} - (S/n)|_1$ the maximum of $|\text{Ric} - (S/n)|$.

Theorem 3.1. *Let M be a compact Riemannian spin n -manifold and let λ be any eigenvalue of the Dirac operator D . Then, for all $t \geq 0$, we have*

$$\lambda^2 \geq \frac{n}{4(n-1)} \frac{\beta(t)}{\alpha(t)}. \tag{43}$$

Proof. By Lemma 2.2 in [5], the inequalities

$$\begin{aligned} -\frac{S_1 - S_0}{4} (\lambda^2 - \frac{S_0}{4}) \int_M |\psi|^2 &\leq \int_M \left(\lambda^2 - \frac{S}{4} \right) \left(|\nabla \psi|^2 - \left(\lambda^2 - \frac{S}{4} \right) |\psi|^2 \right) \\ &\leq \frac{S_1 - S_0}{4} \left(\lambda^2 - \frac{S_0}{4} \right) \int_M |\psi|^2 \end{aligned} \tag{44}$$

are valid for any eigenspinor ψ to the eigenvalue λ of D . Using (31), (36), (41) and (44) we obtain (43) if we integrate Eq. (39). □

We obtain the following corollary by computing the maximum of $\beta(t)$ for $t \geq 0$.

Corollary 3.1. *There are no harmonic spinors on a compact Riemannian spin manifold with $S_0 \leq 0$ if the condition*

$$|\text{Ric}|_0^2 > S_0 \left(\kappa_0 - \frac{S_1 - S_0}{4} \right) + \sqrt{8|S_0|}\vartheta \tag{45}$$

is satisfied. In particular, the kernel of D is trivial if $S_0 = 0$ and $|\text{Ric}|_0 > 0$.

Remark 3.1.

- (i) Our Corollary 3.1 is a generalization of Theorem 2.1 in [2] since, in the case of a harmonic curvature tensor ($\delta R = 0$), we have $dS = 0$ and $\vartheta = 0$.

(ii) The inequality (43) can be written in the form

$$\lambda^2 \geq \frac{n}{4(n-1)} \left(S_0 + t \frac{\gamma(t)}{\alpha(t)} \right), \tag{46}$$

where $\gamma(t)$ is the function given by

$$\begin{aligned} \gamma(t) := & |\text{Ric}|_0^2 - \frac{S_0}{n-1} \left(S_1 - \kappa_0 + \frac{S_1 - S_0}{4} \right) - \kappa_0(S_* - S_0) \\ & - 2t \left(\frac{nS_0}{4(n-1)} \left| \text{Ric} - \frac{S}{n} \right|_1^2 + \vartheta \right). \end{aligned}$$

Thus, for $S_0 > 0$, (46) yields a better estimate than (1) if $\gamma(t) > 0$ for some $t > 0$. We see immediately that this is the case if the condition

$$|\text{Ric}|_0^2 > \frac{S_0}{n-1} \left(S_1 - \kappa_0 + \frac{S_1 - S_0}{4} \right) + \kappa_0(S_* - S_0) \tag{47}$$

is fulfilled. This generalizes a corresponding assertion in [2, Section 2]. In particular, if S is constant and positive, (47) simplifies to

$$|\text{Ric}|_0^2 > \frac{S}{n-1} (S - \kappa_0). \tag{48}$$

(iii) The limiting case of (43) corresponds to the limiting case of (1) since, by the same arguments that we used in Section 2 of [2], it follows that (43) can be an equality for the first eigenvalue of D for $t = 0$ only.

In order to write down the main result of this section the notations:

$$\begin{aligned} A &:= |\text{Ric}|_0^2 - \frac{S_0}{n-1} \left(S_1 - \kappa_0 + \frac{S_1 - S_0}{4} \right) - \kappa_0(S_* - S_0), \\ b &:= \frac{n}{n-1} \left(\frac{S_1}{n} - \kappa_0 + \frac{S_1 - S_0}{4} \right), \quad c := \left| \text{Ric} - \frac{S}{n} \right|_1 \sqrt{\frac{2n}{n-1}}, \\ a &:= \frac{4}{A} \left(\frac{nS_0}{4(n-1)} \left| \text{Ric} - \frac{S}{n} \right|_1^2 + \vartheta \right) \end{aligned}$$

are convenient. The function $t\gamma(t)/\alpha(t)$ attains its maximum for $t > 0$ if the condition (47) is satisfied, i.e., if $A > 0$. By computing this maximum and assertion (iii) of Remark 3.1, we obtain the following result.

Corollary 3.2. *Let M be a compact Riemannian spin n -manifold with $A \geq 0$. Then, for every eigenvalue λ of the Dirac operator, we have the inequality*

$$\lambda^2 \geq \frac{n}{4(n-1)} \left(S_0 + \frac{A}{a + b + \sqrt{a^2 + 2ab + c^2}} \right), \tag{49}$$

which is never an equality if $A > 0$.

Corollary 3.3. *If M is a compact Riemannian spin n -manifold such that $S_0 = 0$ and $|\text{Ric}|_0 > 0$, then every eigenvalue λ of the Dirac operator satisfies the estimate*

$$\lambda^2 > \frac{n}{4(n-1)} \frac{|\text{Ric}|_0^2}{a + b + \sqrt{a^2 + 2ab + c^2}} \tag{50}$$

with the constants

$$a = \frac{4\vartheta}{|\text{Ric}|_0^2}, \quad b = \frac{n}{n-1} \left(\frac{n+4}{4n} S_1 - \kappa_0 \right), \quad c = \left| \text{Ric} - \frac{S}{n} \right|_1 \sqrt{\frac{2n}{n-1}}.$$

Remark 3.2.

- (i) Our Corollary 3.2 is comparable with Theorem 3.1. in [2], which uses the additional assumption that $\delta R = 0$. But Corollary 3.2 is not a direct generalization of this Theorem 3.1. Since the application of Corollary 3.2 to the case of a harmonic curvature tensor yields a weaker result than Theorem 3.1. In particular, applying Corollary 3.3 to the special case of $\delta R = 0$, the estimate (50) may be written as

$$\lambda^2 > \frac{1}{4} \frac{|\text{Ric}|_0^2}{|\kappa_0| + |\text{Ric}|_1 \sqrt{2(n-1)/n}} \tag{51}$$

since $\delta R = 0$ implies $E = 0$ and $dS = 0$ and, hence, $\vartheta = 0$. Comparing (3) and (51) we see that (51) is a weaker estimate than (3).

- (ii) Corollary 4.1 in [5] is a result similar to Corollary 3.3, it was obtained under the additional assumption that the Ricci tensor commutes with its covariant derivatives of first order ($[\text{Ric}, \nabla_X \text{Ric}] = 0$).
- (iii) Examples 4.1. and 4.2. in [5] yield simple examples of manifolds for which the lower bounds in the estimates (49) or (50), respectively, can be computed easily.

4. Weyl tensor depending estimates

Our estimate (49) cannot be better than (1) if M is Einstein or if $|\text{Ric}|_0 = 0$. In this section we prove estimates that also work in such situations. For $s, t \in \mathbb{R}$, let

$$\mathcal{R}^s, \mathcal{S}^t : \Gamma(\Sigma) \rightarrow \Gamma(TM \otimes \Sigma)$$

be the first-order differential operators locally defined by $\mathcal{R}^s \psi := X^k \otimes \mathcal{R}_{X_k}^s \psi$, $\mathcal{S}^t \psi = X^k \otimes \mathcal{S}_{X_k}^t \psi$ with

$$\mathcal{R}_X^s \psi := \mathcal{D}_X \psi - s \delta B(X) \psi, \quad \mathcal{S}_X^t \psi := \mathcal{D}_X \psi - t B(X, X^k) \nabla_{X_k} \psi.$$

Then, for any $\psi \in \Gamma(\Sigma)$, we have

$$|\mathcal{R}^s \psi|^2 = |\mathcal{D} \psi|^2 + 2s \text{Re} \langle \delta B(X^k) \nabla_{X_k} \psi, \psi \rangle + s^2 \langle F \psi, \psi \rangle, \tag{52}$$

$$\begin{aligned} |\mathcal{S}^t \psi|^2 &= |\mathcal{D} \psi|^2 - 2t \text{Re} \langle \delta B(X^k) \nabla_{X_k} \psi, \psi \rangle - t \langle H \psi, \psi \rangle \\ &\quad + 2t \text{div} (\text{Re} \langle B(X^k, X^l) \nabla_{X_l} \psi, \psi \rangle X_k) + t^2 \langle B^2(X^k, X^l) \nabla_{X_k} \psi, \nabla_{X_l} \psi \rangle \end{aligned} \tag{53}$$

and, hence,

$$\begin{aligned} \frac{1}{2}(|\mathcal{R}^{2t}\psi|^2 + |\mathcal{S}^{2t}\psi|^2) &= |\mathcal{D}\psi|^2 - t\langle H\psi, \psi \rangle + 2t \operatorname{div}(\operatorname{Re}\langle B(X^k, X^l)\nabla_{X_l}\psi, \psi \rangle X_k) \\ &\quad + 2t^2(\langle F\psi, \psi \rangle + \langle B^2(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi \rangle). \end{aligned} \quad (54)$$

Theorem 4.1. *Let M be a compact Riemannian spin n -manifold with harmonic Weyl tensor ($\delta W = 0$) and let λ be any eigenvalue of the Dirac operator. Then, for all $t \geq 0$, the inequality*

$$\lambda^2 \geq \frac{n}{4(n-1)} \left(S_0 + \frac{4\nu_0 t - (n-1)\mu^2 S_0 t^2}{1 + n(n-1)\mu^2 t^2} \right) \quad (55)$$

is valid.

Proof. By (21), $\delta W = 0$ implies $\delta B = 0$. Integrating Eq. (53) for any eigenspinor ψ ($D\psi = \lambda\psi$) we find (55) by using $\delta B = 0$, (30), (31), (33) and (36). \square

The following result is proved by computing the maximum of the right-hand side of (55) for $t \geq 0$.

Corollary 4.1. *Let M be a compact Riemannian spin n -manifold with $\delta W = 0$ and $\mu > 0$. Then every eigenvalue λ of the Dirac operator satisfies the estimate*

$$\lambda^2 \geq \frac{1}{8(n-1)} \left((2n-1)S_0 + \sqrt{S_0^2 + \frac{n}{n-1} \left(\frac{4\nu_0}{\mu} \right)^2} \right). \quad (56)$$

For $S_0 \leq 0$, this lower bound is positive if

$$\nu_0 > \frac{1}{2}(n-1)|S_0|\mu. \quad (57)$$

In particular, there are no harmonic spinors if $S_0 = 0$ and $\nu_0 > 0$.

Every Einstein manifold fulfils the condition $\delta W = 0$. Thus, we obtain the following corollary.

Corollary 4.2. *The estimate (56) is valid on any compact Einstein spin manifold with $\mu > 0$.*

Remark 4.1.

- (i) Comparing (7) and (56) we see that (56) is the better estimate. Thus, our Corollary 4.1 improves Theorem 3.1 in [3].
- (ii) For $S_0 > 0$, (56) yields a better estimate than (1) if $\nu_0 > 0$. By Corollary 4.2, this is also the case if the manifold is Einstein or even Ricci flat.

Our next aim is to prove an estimate similar to (56) for manifolds whose Weyl tensor is not harmonic. We denote by η the supremum of all eigenvalues of the endomorphism F

on Σ . Then $\eta \geq 0$ and it holds that

$$\langle F\psi, \psi \rangle \leq \eta |\psi|^2 \tag{58}$$

for all $\psi \in \Gamma(\Sigma)$.

Theorem 4.2. *Let M be any compact Riemannian spin n -manifold and let λ be any eigenvalue of the Dirac operator. Then, for all $t \geq 0$, we have the inequality*

$$\lambda^2 \geq \frac{n}{4(n-1)} \left(S_0 + \frac{4v_0t - 2((n-1)\mu^2 S_0 + 4\eta)t^2}{1 + 2n(n-1)\mu^2 t^2} \right). \tag{59}$$

Proof. Using (58) we integrate Eq. (54) and find (59) by simple estimates as before. \square

By computing the maximum of the right-hand side of (59) with respect to $t \geq 0$, we obtain the following result.

Corollary 4.3. *If M is a compact Riemannian spin n -manifold with $\mu > 0$, then, for every eigenvalue λ of the Dirac operator D , the estimate*

$$\lambda^2 \geq \frac{1}{8(n-1)} \left((2n-1)S_0 - \frac{4\eta}{(n-1)\mu^2} + \sqrt{\left(S_0 + \frac{4\eta}{(n-1)\mu^2} \right)^2 + \frac{8n}{n-1} \left(\frac{v_0}{\mu} \right)^2} \right) \tag{60}$$

is valid. For $S_0 \leq 0$, this lower bound is positive and, hence, $\ker(D) = 0$ if the condition

$$v_0 > \sqrt{|S_0|(2\eta + \frac{1}{2}(n-1)^2\mu^2|S_0|)} \tag{61}$$

is fulfilled.

Corollary 4.4. *For every eigenvalue λ of the Dirac operator on a compact Riemannian spin n -manifold with $S_0 = 0$ and $v_0 > 0$, we have the estimate*

$$\lambda^2 \geq \frac{n}{4(n-1)} \frac{v_0^2}{\eta + \sqrt{\eta^2 + \binom{n}{2} \mu^2 v_0^2}}. \tag{62}$$

In particular, there are no harmonic spinors.

Remark 4.2.

- (i) For $S_0 > 0$, (60) also yields a better estimate than (1) if $v_0 > 0$.
- (ii) It is not known if there exist manifolds with the property that (56) or (60), respectively, is an equality for the first eigenvalue λ_1 of the Dirac operator.

5. Estimates depending on the whole curvature tensor

In order to obtain estimates for the first eigenvalue of the Dirac operator that depend on the Ricci tensor and also on the Weyl tensor we consider, for all $t \in \mathbb{R}$, the first-order differential operator

$$\mathcal{T}^t : \Gamma(\Sigma) \rightarrow \Gamma(TM \otimes \Sigma),$$

which is locally defined by $\mathcal{T}^t \psi := X^k \otimes \mathcal{T}_{X^k}^t \psi$, and

$$\mathcal{T}_{X^k}^t \psi := \mathcal{D}_X \psi - tC(X, X^k) \nabla_{X^k} \psi.$$

Then, for any $\psi \in \Gamma(\Sigma)$, it holds that

$$\begin{aligned} |\mathcal{T}^t \psi|^2 &= |\mathcal{D}\psi|^2 + \frac{t}{n} \text{Re} \langle \text{Ric}(X^k) \nabla_{X^k} \psi, \mathcal{D}\psi \rangle - 2t \text{Re} \langle \delta C(X^k) \nabla_{X^k} \psi, \psi \rangle - t \langle G\psi, \psi \rangle \\ &\quad - 2t \text{div}(\text{Re} \langle C(X^k, X^l) \nabla_{X_l} \psi, \psi \rangle X_k) + t^2 \langle C^2(X^k, X^l) \nabla_{X^k} \psi, \nabla_{X_l} \psi \rangle. \end{aligned} \tag{63}$$

Lemma 5.1. *Let M be a Riemannian spin n -manifold and let λ be any eigenvalue of the Dirac operator D . Then, for any corresponding eigenspinor $\psi(D\psi = \lambda\psi)$ and all $t \in \mathbb{R}$, we have the equations*

$$\begin{aligned} |\mathcal{T}^t \psi|^2 &= |\mathcal{D}\psi|^2 - t \frac{2n-1}{n} \text{Re} \langle \delta C(X^k) \nabla_{X^k} \psi, \psi \rangle \\ &\quad + \frac{t}{n} \left(\left(\lambda^2 - \frac{S}{4} \right) \left(|\nabla\psi|^2 - \left(\lambda^2 - \frac{S}{4} \right) |\psi|^2 \right) + \langle \nabla_{\text{Ric}(X_k)} \psi, \nabla_{X^k} \psi \rangle \right) \\ &\quad - t \left(\langle H\psi, \psi \rangle + \frac{1}{4n} \left(\frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|^2 + \frac{S^2}{n(n-1)} \right) |\psi|^2 \right) \\ &\quad + \text{div} \left(\frac{t}{n} X_\psi + 2t \text{Re} \langle C(X^k, X^l) \nabla_{X_l} \psi, \psi \rangle X_k \right) \\ &\quad + t^2 \langle C^2(X^k, X^l) \nabla_{X^k} \psi, \nabla_{X_l} \psi \rangle, \end{aligned} \tag{64}$$

$$\begin{aligned} &\frac{1}{2} (|\mathcal{P}^{(2n-1)/n} \psi|^2 + |\mathcal{T}^{2t} \psi|^2) \\ &= |\mathcal{D}\psi|^2 + \frac{t}{n} \left(\left(\lambda^2 - \frac{S}{4} \right) \left(|\nabla\psi|^2 - \left(\lambda^2 - \frac{S}{4} \right) |\psi|^2 \right) + \langle \nabla_{\text{Ric}(X_k)} \psi, \nabla_{X^k} \psi \rangle \right) \\ &\quad - t \left(\langle H\psi, \psi \rangle + \frac{1}{4n} \left(\frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|^2 + \frac{S^2}{n(n-1)} \right) |\psi|^2 \right) \\ &\quad - \text{div} \left(\frac{t}{n} X_\psi + 2t \text{Re} \langle C(X^k, X^l) \nabla_{X_l} \psi, \psi \rangle X_k \right) \\ &\quad + 2t^2 \left(\langle C^2(X^k, X^l) \nabla_{X^k} \psi, \nabla_{X_l} \psi \rangle + \left(\frac{2n-1}{2n} \right)^2 \langle \mathcal{E}\psi, \psi \rangle \right). \end{aligned} \tag{65}$$

Proof. Inserting (19) and (40) into (63) we find (64). Using (37) and (64) we obtain (65). □

Again, let M be compact. By $|S|_0$ we denote the minimum of the function $|S|$ on M and we use the notation

$$S_\star := \begin{cases} S_0 & \text{if } \kappa \geq 0, \\ S_1 & \text{if } \kappa < 0. \end{cases}$$

Moreover, we introduce six functions $\alpha_p, \beta_p, \gamma_p : \mathbb{R} \rightarrow \mathbb{R}, p \in \{1, 2\}$, defined by

$$\begin{aligned} \alpha_p(t) &:= 1 + \frac{1}{n-1} \left(\kappa + \frac{S_1 - S_0}{4} \right) + pn(n-1)\xi^2 t^2, \\ \beta_p(t) &:= S_0 + t \left(4v_0 + \frac{1}{n} \left(\frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{|S|_0^2}{n(n-1)} + \frac{S_0(S_1 - S_0)}{4} + S_\star \kappa \right) \right) \\ &\quad + pt^2 \left((n-1)^2 S_0 \xi^2 - \left(\frac{2n-1}{n} \right)^2 \vartheta \right), \\ \gamma_p(t) &:= 4v_0 + \frac{1}{n} \left(\frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{|S|_0^2}{n(n-1)} \right. \\ &\quad \left. - \frac{S_0}{n-1} \left(\kappa + \frac{S_1 - S_0}{4} \right) + \kappa(S_\star - S_0) \right) \\ &\quad - pt \left((n-1) S_0 \xi^2 + \left(\frac{2n-1}{n} \right)^2 \vartheta \right). \end{aligned}$$

Theorem 5.1. *Let λ be any eigenvalue of the Dirac operator on a compact Riemannian spin n -manifold. Then the following holds:*

(i) *For any $t \geq 0$ with $\beta_2(t) > 0$, we have the estimate*

$$\lambda^2 \geq \frac{n}{4(n-1)} \frac{\beta_2(t)}{\alpha_2(t)} = \frac{n}{4(n-1)} \left(S_0 + t \frac{\gamma_2(t)}{\alpha_2(t)} \right). \tag{66}$$

(ii) *If the curvature tensor is harmonic, then the estimate*

$$\lambda^2 \geq \frac{n}{4(n-1)} \frac{\beta_1(t)}{\alpha_1(t)} = \frac{n}{4(n-1)} \left(S + t \frac{\gamma_1(t)}{\alpha_1(t)} \right) \tag{67}$$

is valid for every $t \geq 0$ with $\beta_1(t) > 0$.

Proof. Integrating Eq. (65) and using (33), (41) and (42), for any $t \geq 0$, we obtain

$$\lambda^2 \alpha_2(t) \geq \frac{n}{4(n-1)} \beta_2(t) = \frac{n}{4(n-1)} (S_0 \alpha_2(t) + \gamma_2(t)). \tag{68}$$

In particular, (68) shows that $\beta_2(t) > 0$ ($t \geq 0$) forces $\alpha_2(t) > 0$. This proves the assertion (i) of our theorem. Further, the supposition $\delta R = 0$ implies $\delta C = 0$ by (21) and,

moreover, $\vartheta = 0$, $S_0 = S_1 = S$. Thus, integrating Eq. (64) one analogously proves the assertion (ii). \square

Corollary 5.1. *On a compact Riemannian spin n -manifold with $S_0 \leq 0$, we have the following:*

(i) *There are no harmonic spinors if the condition*

$$4nv_0 + \frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{|S|_0^2}{n(n-1)} + S_{\star} \kappa > \frac{|S_0|(S_1 - S_0)}{4} + 4 \cdot \sqrt{2|S_0| \left(\binom{n}{2}^2 |S_0| \zeta^2 + \left(\frac{2n-1}{2} \right)^2 \vartheta \right)} \tag{69}$$

is satisfied. In particular, for $S_0 = 0$, there are no harmonic spinors if $v_0 > 0$ or $|\text{Ric} - (S/n)|_0 > 0$.

(ii) *If the curvature tensor is harmonic, then there exist no harmonic spinors if*

$$4nv_0 + \frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{S^2}{n(n-1)} > |S| \left(\kappa + 4 \binom{n}{2} \zeta \right). \tag{70}$$

In particular, for $S = 0$, we have $\ker(D) = 0$ if $v_0 > 0$ or $|\text{Ric}|_0 > 0$.

Proof. (69) implies that the function $\beta_2(t)$ attains positive values for some $t > 0$. The condition (70) implies that also the function $\beta_1(t)$ has this property. \square

Remark 5.1.

(i) If the condition

$$4nv_0 + \frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{|S|_0^2}{n(n-1)} > \frac{S_0}{n-1} \left(\kappa + \frac{S_1 - S_0}{4} \right) \tag{71}$$

is satisfied on a compact Riemannian spin n -manifold with $S_0 > 0$, then (66) yields a better estimate than (1) since this condition implies that the function $\gamma_2(t)$ attains positive values for some $t > 0$. We note that $S_0 > 0$ implies $\kappa > 0$ and, hence, $\alpha_2(t) \geq 1$ for $t \geq 0$.

(ii) In the case of a harmonic curvature tensor, the function $\gamma_1(t)$ reaches positive values for some $t > 0$ if

$$4nv_0 + \frac{n+2}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 > \frac{S}{n-1} \left(\kappa - \frac{S}{n} \right). \tag{72}$$

Thus, if $S > 0$ and (72) is fulfilled, (67) yields a better estimate than (1).

(iii) The assertion (ii) of Corollary 5.1 is an improvement of Theorem 4.1 in [3], where, instead of ζ , another curvature invariant σ was used. ζ and σ are related by

$$\zeta \leq \frac{1}{2} \binom{n}{2} \sigma \tag{73}$$

(see [3, Section 4]). Replacing ζ by the value

$$\frac{1}{2} \binom{n}{2} \sigma$$

inequality (70) becomes a condition that is weaker than the condition (38) in [3].

In the end of this paper we show that another combination of our basic Weitzenböck formulas leads to similar results, but they do not contain the curvature invariants κ_0 and κ . Using (37), (38) and (63) we find the equation

$$\begin{aligned} & \frac{1}{2} (|\mathcal{Q}^{t/n} \psi|^2 + |\mathcal{T}^{2t} \psi|^2) \\ &= |\mathcal{D}\psi|^2 + t \frac{S}{n} |D\psi|^2 - t \langle G\psi, \psi \rangle - 2t \operatorname{Re} \langle \delta C(X^k) \nabla_{X_k} \psi, \psi \rangle \\ & \quad - 2t \operatorname{div}(\operatorname{Re} \langle C(X^k, X^l) \nabla_{X_l} \psi, \psi \rangle X_k) \\ & \quad + 2t^2 \left(\frac{1}{4n^2} \left| \operatorname{Ric} - \frac{S}{n} \right|^2 |D\psi|^2 + \langle C^2(X^k, X^l) \nabla_{X_k} \psi, \nabla_{X_l} \psi \rangle \right) \end{aligned} \tag{74}$$

and, moreover,

$$\begin{aligned} & \frac{1}{3} (|\mathcal{P}^{3t} \psi|^2 + |\mathcal{Q}^{3t/2n} \psi|^2 + |\mathcal{T}^{3t} \psi|^2) \\ &= |\mathcal{D}\psi|^2 - t \langle G\psi, \psi \rangle + t \frac{S}{n^2} |D\psi|^2 + \frac{t}{2n} \operatorname{Re} \langle D\psi, dS \cdot \psi \rangle \\ & \quad - 2t \operatorname{div}(\operatorname{Re} \langle C(X^k, X^l) \nabla_{X_l} \psi, \psi \rangle X_k) \\ & \quad + 3t^2 \left(\langle \mathcal{E}\psi, \psi \rangle + \frac{1}{4n^2} \left| \operatorname{Ric} - \frac{S}{n} \right|^2 |D\psi|^2 + \langle C^2(X^k, X^l) \nabla_{X_k} \psi, \nabla_{X_l} \psi \rangle \right). \end{aligned} \tag{75}$$

Both equations are valid for any $t \in \mathbb{R}$ and any $\psi \in \Gamma(\Sigma)$. We introduce the six functions $\alpha_p, \beta_p, \gamma_p : \mathbb{R} \rightarrow \mathbb{R}, p \in \{3, 4\}$, defined by

$$\begin{aligned} \alpha_p(t) &:= 1 + t \frac{S_1}{n(n-1)} + (p-1)t^2 \left(\frac{1}{4n(n-1)} \left| \operatorname{Ric} - \frac{S}{n} \right|_1^2 + n(n-1)\zeta^2 \right), \\ \beta_p(t) &:= S_0 + t \left(4v_0 + \frac{2}{n-2} \left| \operatorname{Ric} - \frac{S}{n} \right|_0^2 + \frac{|S|_0^2}{n(n-1)} \right) + (p-1)t^2 ((n-1)^2 S_0 \zeta^2 - 4\vartheta), \\ \gamma_p(t) &:= 4v_0 + \frac{2}{n-2} \left| \operatorname{Ric} - \frac{S}{n} \right|_0^2 - \frac{S_0}{n(n-1)} (S_1 - S_0) \\ & \quad - (p-1)tS_0 \left(\frac{1}{4n(n-1)} \left| \operatorname{Ric} - \frac{S}{n} \right|_1^2 + (n-1)\zeta^2 + 4\vartheta \right). \end{aligned}$$

Theorem 5.2. *Let λ be any eigenvalue of the Dirac operator on a compact Riemannian spin n -manifold. Then the following holds:*

(i) For every $t \geq 0$ with $\beta_4(t) > 0$, we have the estimate

$$\lambda^2 \geq \frac{n}{4(n-1)} \frac{\beta_4(t)}{\alpha_4(t)} = \frac{n}{4(n-1)} \left(S_0 + t \frac{\gamma_4(t)}{\alpha_4(t)} \right). \tag{76}$$

(ii) In the special case that $\delta R = 0$, the estimate

$$\lambda^2 \geq \frac{n}{4(n-1)} \frac{\beta_3(t)}{\alpha_3(t)} = \frac{n}{4(n-1)} \left(S + t \frac{\gamma_3(t)}{\alpha_3(t)} \right) \tag{77}$$

is valid for every $t \geq 0$ with $\beta_3(t) > 0$.

Proof. Inserting any eigenspinor ψ to the eigenvalue λ of D into Eq. (75) and then integrating this equation we obtain (76) by (19), (32) and (41) and analogous considerations as in the proof of Theorem 5.1. In the special case of $\delta R = 0$, we integrate equation (74) for any eigenspinor ψ . Then we find (77). \square

Studying the conditions under which the functions $\beta_3(t)$ and $\beta_4(t)$, respectively, attain positive values for some $t > 0$, we immediately obtain the next result.

Corollary 5.2. *The following holds on a compact Riemannian spin n -manifold with $S_0 \leq 0$:*

(i) There are no harmonic spinors if

$$4n\nu_0 + \frac{2n}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{|S|_0^2}{n-1} > 4 \cdot \sqrt{3|S_0| \left(\binom{n}{2}^2 |S_0|\zeta^2 + n^2\vartheta \right)}. \tag{78}$$

In particular, for $S_0 = 0$, we have $\ker(D) = 0$ if $\nu_0 > 0$ or $|\text{Ric} - (S/n)|_0 > 0$.

(ii) In the special situation that $\delta R = 0$, there are no harmonic spinors if

$$4n\nu_0 + \frac{2n}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 + \frac{S^2}{n-1} > 4 \binom{n}{2} \zeta |S| \sqrt{2}. \tag{79}$$

Remark 5.2.

(i) For $S_0 > 0$, (76) gives a better estimate than (1) if

$$4n\nu_0 + \frac{2n}{n-2} \left| \text{Ric} - \frac{S}{n} \right|_0^2 > \frac{S_0}{n-1} (S_1 - S_0). \tag{80}$$

(ii) In the special case of a harmonic curvature tensor and $S > 0$, (77) yields a better estimate than (1) if $\nu_0 > 0$ or $|\text{Ric} - (S/n)|_0 > 0$.

(iii) The same arguments that are used in the proof of Theorem 4.2 in [3] show that, for an optimal parameter $t_0 > 0$, the inequalities (66), (67) and (76), (77) can never be equalities for the first eigenvalue of the Dirac operator.

(iv) If the first-order covariant derivatives of the Ricci tensor commute ($[\nabla_X \text{Ric}, \nabla_Y \text{Ric}] = 0$), we see, by (29), that the number ϑ , which enters the estimates (43), (66) and (76), is simply the maximum of the function $(1/4)|\nabla \text{Ric}|^2 - ((n+1)/16n)|dS|^2$. Moreover, in

this case it becomes obvious, owing to (28) that the number η , which occurs in Section 4, is given by the maximum of the function $(1/4)|\nabla\text{Ric}|^2 - (n/16(n-1))|dS|^2$ then.

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